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PARTIALLY SINGULAR CONTROL PROBLEMS AS SINGULAR SINGULAR-PERTURBATION PROBLEMS

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Abstract. This paper demonstrates how solutions to partially singular control problems can be obtained as the limit of nearly singular problems. The asymptotic analysis involved relates to a developing theory of singular singular-perturbation problems and is of independent practical interest. The results obtained are related to other current literature.

Keywords. Singular control; singular perturbations; asymptotic methods.

GENERAL DESCRIPTION

Now-classical linear control theory completely solves the problem of minimizing the cost function

$$J = \frac{1}{2} x'(1)\Pi x(1) + \frac{1}{2}\int_0^1 [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt \tag{1}$$

subject to the state equation

$$\dot{x} = A(t)x + B(t)u, \quad x(0) \text{ prescribed, on } 0 \le t \le 1 \tag{2}$$

for symmetric matrices Π, Q, and R which are respectively positive semidefinite, semidefinite, and definite matrices [cf. Kalman (1960) or, e.g., Anderson and Moore (1971)]. We shall consider the singular problem where R is a singular matrix, and shall take x to be an n vector and u to be an m vector. Though such singular problems are important for applications, their solutions are not readily obtained [cf. Bell and Jacobson (1975)]. Jameson and this author [cf. O'Malley and Jameson (1975, 1977)] obtained very satisfactory solutions to certain totally singular problems ($R \equiv 0$), by considering them as the limit of nearly singular problems with $R = \epsilon^2 R$, R nonsingular, as the small positive parameter ϵ tends to zero. This idea was used earlier in the singular control literature by Jacobson and coworkers to both theoretical and numerical advantage [cf. Jacobson and Speyer (1970) and Jacobson et al. (1970)] and is analogous to the common use of artificial viscosity in the fluid dynamics literature [cf., e.g., Richtmyer and Morton (1967) and Roache (1972)]. We note that problems with singular or nearly singular matrices R also occur naturally in deterministic, stochastic, and distributed parameter problems [cf. Kwakernaak and Sivan (1972), Friedland (1971), and Lions (1973)].

We'll study the partially singular situation where the matrix R is singular and of constant nonzero rank, by considering the asymptotic solution of the nearby nonsingular problem with

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & \epsilon^2 R_2 \end{bmatrix} \equiv \text{diag}\,(R_1, \epsilon^2 R_2) \tag{3}$$

where R_1 and R_2 are nonsingular $m_1 \times m_1$ and $m_2 \times m_2$ matrices with $m_1 + m_2 = m$. Related approaches to partially singular problems have been previously analyzed by Womble et al. (1976) and Hutton (1974). Since the last m_2 components of the control are "cheap" compared to the first m_1 components, we might expect those components of the optimal control to involve large initial impulses.

The classical Kalman theory implies that the optimal control for the linear regulator problem (1)-(2) is given by a feedback control law

$$u = -R^{-1}B'kx \tag{4}$$

where k is the unique symmetric positive semidefinite solution of the $n \times n$ matrix Riccati terminal value problem

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$$\dot{k} + kA + A'k + Q = kBR^{-1}B'k, \quad k(1) = \Pi. \tag{5}$$

Rewriting

$$B = [B_1 \; B_2] \tag{6}$$

to conform with the partitioning (3) of R, we obtain the singularly perturbed Riccati equation

$$\epsilon^2(\dot{k} + kA + A'k + Q - kB_1R_1^{-1}B_1'k) = kB_2R_2^{-1}B_2'k, \qquad k(1) = \Pi. \tag{7}$$

In most singular perturbation problems, the limiting solution as the small parameter ϵ goes to zero is usually obtained as the solution of the reduced problem obtained when $\epsilon = 0$ [cf. O'Malley (1974) and (1978a)]. Here, then, k might be expected to have an asymptotic expansion

$$K(t,\epsilon) \sim \sum_{j=0}^{\infty} K_j(t)\epsilon^j \tag{8}$$

for $t < 1$. Corresponding to (7), we have the reduced equation

$$K_0B_2R_2^{-1}B_2'K_0 = 0. \tag{9}$$

Any solution K_0 of (9) must satisfy

$$B_2'K_0 = 0 \tag{10}$$

for the $m_2 \times n$ matrix B_2'. In the unlikely situation that B_2 were square and nonsingular, we'd obtain the unique solution $K_0 = 0$. Otherwise, (9) merely restricts K_0 to lie in the null space of B_2', a space of dimension complementary to the rank of B_2. We note that (8) will not generally be valid near $t = 1$ because the given matrix Π may not satisfy $B_2'(1)\Pi = 0$, in contradiction to (7) and (10). Our experience suggests that the solution k of (7) will often be of the form

$$k(t,\epsilon) = K(t,\epsilon) + \ell(\sigma,\epsilon) \tag{11}$$

where the "outer solution" K and the "boundary layer correction" ℓ both have asymptotic power series expansions in ϵ and the terms of ℓ tend to zero as the stretched variable

$$\sigma = (1 - t)/\epsilon \tag{12}$$

tends to infinity. For $t < 1$, the asymptotic solution will be provided by the outer expansion (8). To obtain k in the simple form (11), we'll later need to ask that Π_0, Π when $\epsilon = 0$, is zero.

Using the control law (4), we find that the corresponding trajectory must satisfy the linear system

$$\dot{x} = (A - BR^{-1}B'k)x, \quad x(0) \text{ prescribed}. \tag{13}$$

Thus, it also satisfies the singularly perturbed initial value problem

$$\epsilon^2\dot{x} = \epsilon^2(A - B_1R_1^{-1}B_1'k)x - B_2R_2^{-1}B_2'kx, \qquad x(0) \text{ prescribed}. \tag{14}$$

For $0 < t < 1$, then, we can expect the limiting trajectory to satisfy the reduced equation

$$B_2R_2^{-1}B_2'K_1X_0 = 0 \tag{15}$$

(since (10) implies that $B_2'k \sim \epsilon B_2'K_1$ there) and this implies that we must satisfy the singular arc condition

$$B_2'K_1X_0 = 0. \tag{16}$$

Here, we'll obtain the unique solution $X_0 = 0$ if the $m_2 \times n$ matrix $B_2'K_1$ happens to be square and nonsingular. Generally, however, X_0 is merely restricted to lie in the null space of $B_2'K_1$. We find that there is usually nonuniform convergence at $t = 0$, because the prescribed initial vector won't generally satisfy $B_2'(0)K_1(0)x(0) = 0$. A boundary layer at $t = 1$ is also expected since k converges nonuniformly there. Thus, the trajectory might be asymptotically of the form

$$x(t,\epsilon) = X(t,\epsilon) + m(\tau,\epsilon) + n(\sigma,\epsilon) \tag{17}$$

where the outer solution X is asymptotically valid for $0 < t < 1$ and the boundary layer correction m tends to zero as the stretched variable

$$\tau = t/\epsilon \tag{18}$$

tends to infinity while the other boundary layer correction $n \to 0$ as $\sigma \to \infty$.

Both the problems (7) and (14) for the Riccati gain k and the state x are generally singular singular-perturbation problems. For each $\epsilon > 0$, they have a unique solution. The limiting problems (10) and (16), however, generally have an infinity of solutions (since the matrices B_2' and $B_2'K_1$ have nontrivial null spaces) and further less obvious analysis is required to obtain the unique limiting solutions K_0 and X_0 within $0 < t < 1$. Such problems are of increasing importance generally in singular perturbations theory and in control [cf. O'Malley (1977), O'Malley and Flaherty (1977), Utkin (1978), or Vasil'eva (1975)]. Other approaches toward their solution might include more explicit use of the matrix pseudoinverse [cf., e.g., Campbell (1976) and Campbell et al. (1976)] or, for time-invariant problems, frequency domain techniques [cf. Francis (1977) and Francis and Glover (1977)]. We note that standard numerical methods for stiff initial value problems will generally be unable to

integrate (7) and (14) [cf. Flaherty and O'Malley (1978)].

We shall limit attention here to the Riccati problem (7) under the hypothesis that

$$B_2'QB_2 > 0, \tag{19}$$

i.e., this $m_2 \times m_2$ matrix is positive definite. For $m_1 = 0$, this restriction corresponds to singular arcs of order one and to initial delta function impulses in the optimal control [cf. O'Malley and Jameson (1975, 1977) for some discussion and motivation]. As is usual in singular perturbations analysis, we will separately examine the outer solution $K(t,\varepsilon)$ and the boundary layer correction $\ell(\sigma,\varepsilon)$ [cf. equation (11)].

THE OUTER RICCATI GAIN

The outer solution $K(t,\varepsilon)$ must asymptotically represent the solution to (7) for $t < 1$, so we must have

$$\varepsilon^2(\dot{K} + KA + A'K + Q - KB_1R_1^{-1}B_1'K) = KB_2R_2^{-1}B_2'K \tag{20}$$

satisfied as a power series (8) in ε. In particular, when $\varepsilon = 0$, we obtain (10). Postmultiplying by B_2, we have an equation for $(KB_2)^{\cdot}$. Premultiplying it by B_2' and equating coefficients of ε^2 then implies that

$$B_2'K_1B_2R_2^{-1}B_2'K_1B_2 = B_2'QB_2 > 0 \tag{21}$$

by (19), and this allows us to solve (20) for $B_2'KB_2 \sim \varepsilon B_2'K_1B_2 > 0$. We then obtain

$$B_2'K = \varepsilon^2R_2(B_2'KB_2)^{-1}[B_{21}'K + B_2'Q + (B_2'K)^{\cdot} + B_2'KA - B_2'KB_1R_1^{-1}B_1'K] \tag{22}$$

where $B_{21} = AB_2 - \dot{B}_2$. Substituting back into (20) finally yields

$$\begin{aligned}\dot{K} + KA + A'K + Q - KB_1R_1^{-1}B_1'K \\ = [KB_{21} + QB_2 + (KB_2)^{\cdot} + A'KB_2 \\ - KB_1R_1^{-1}B_1'KB_2][B_2'QB_2 + B_2'(KB_2)^{\cdot} + B_2'KB_{21} \\ + B_2'A'KB_2 - B_2KB_1R_1^{-1}B_1'KB_2]^{-1}[KB_{21} + QB_2 \\ + (KB_2)^{\cdot} + A'KB_2 - KB_1R_1^{-1}B_1'KB_2]'.\end{aligned} \tag{23}$$

Differential equations for successive terms K_j in the outer expansion (8) for K now follow by equating coefficients termwise in (22) and (23).

When $\varepsilon = 0$, then, we get the parameter free Riccati equation

$$\dot{K}_0 + K_0A_1 + A_1'K_0 + Q_1 = K_0S_1K_0 \tag{24}$$

subject to $B_2'K_0 = 0$ where

$$A_1 = A - B_{21}(B_2'QB_2)^{-1}B_2'Q,$$

and

$$Q_1 = Q - QB_2(B_2'QB_2)^{-1}B_2'Q,$$

and

$$S_1 = [B_1\ B_{21}](\mathrm{diag}(R_1,\ B_2'QB_2))^{-1}[B_1\ B_{21}]'.$$

Standard linear regulator theory implies that a unique positive semidefinite solution to (24) exists when a positive semidefinite terminal value $K_0(1) \geq 0$ is prescribed since $Q_1 = P_1'QP_1 \geq 0$ where

$$P_1 = I - B_2(B_2'QB_2)^{-1}B_2'Q \tag{25}$$

[cf. Anderson and Moore (1971)]. Moreover, $B_2'K_0$ remains constant along solutions of (24). Successive terms K_j satisfy linearized versions of (24) with $B_2'K_j$ obtained termwise from (22).

We note that (25) implies that

$$I = P_1 + P_2 \quad \text{for} \quad P_2 = B_2(B_2'QB_2)^{-1}B_2'Q. \tag{26}$$

Indeed P_1 and P_2 are projections ($P_1^2 = P_1$ and $P_2^2 = P_2$) such that

$$B_2'QP_1 = 0, \quad P_1B_2 = 0, \quad \text{and} \quad P_1P_2 = 0.$$

Thus, P_1 maps into $N(B_2'Q)$, the null space of $B_2'Q$, and P_2 maps into $R(B_2)$, the range of B_2. Likewise, P_1' maps into $N(B_2')$ and P_2' into $R(QB_2)$. [In the special case that P_1 is symmetric, $P_1'P_2 = 0$ and (26) provides a direct sum decomposition of n-space.] Here (10) implies that

$$P_2'K_0 = 0, \tag{28}$$

so the symmetric matrix K_0 satisfies

$$K_0 = K_0P_1 = P_1'K_0 = P_1'K_0P_1.$$

Thus (24) becomes a terminal value problem for $P_1'K_0$. It would seem natural to solve it subject to the boundary value

$$\begin{aligned}K_0(1) &= P_1'(1)K_0(1)P_1(1) \\ &= P_1'(1)\Pi_0P_1(1) \geq 0.\end{aligned} \tag{29}$$

The Riccati equation (24) and the terminal value (29) are the additional data needed to supplement the reduced equation (10) in

order to uniquely determine the limiting outer Riccati gain K_0 for $t < 1$. Since P_1 is singular, we note that (24) is essentially a lower order Riccati equation for $P_1'K_0P_1$ in the null space of B_2' [cf. the analogous discussion for an algebraic Riccati equation in Kwatny (1974) and those for the Riccati differential equation in Friedland (1971) and Moylan and Moore (1971)].

For higher order terms K_j, (22) implies an algebraic equation for $P_2'K_j$. In order to uniquely obtain the outer Riccati gain $K(t,\epsilon)$, then, we need only prescribe a terminal matrix

$$P_1'(1)K(1,\epsilon)P_1(1). \tag{30}$$

Splitting such a problem up into an algebraic equation for $P_2'K$ and then a differential equation for $P_1'K$ corresponds to the auxiliary and bifurcation equations common in bifurcation theory [cf. Cesari (1975)]. This approach can be shown to be more generally tenable for singular problems [cf. O'Malley (1978b)].

THE BOUNDARY LAYER CORRECTION FOR k

Because the Riccati gain k and its outer solution K both satisfy (7), the presumed representation (11) implies that the boundary layer correction $\ell(\sigma,\epsilon)$ must satisfy

$$\begin{aligned} -\epsilon\ell_\sigma &+ \epsilon^2[\ell A + A'\ell - \ell B_1R_1^{-1}B_1'\ell \\ &- KB_1R_1^{-1}B_1'\ell - \ell B_1R_1^{-1}B_1'K] \\ &= \ell B_2R_2^{-1}B_2'\ell + \ell B_2R_2^{-1}B_2'K + KB_2R_2^{-1}B_2'\ell. \end{aligned} \tag{31}$$

If we seek a power series solution

$$\ell(\sigma,\epsilon) \sim \sum_{j=0}^{\infty} \ell_j(\sigma)\epsilon^j \tag{32}$$

to (31), the limiting equation obtained when $\epsilon = 0$ implies that the leading coefficient ℓ_0 must satisfy

$$B_2'\ell_0 = 0. \tag{33}$$

Together, (10) and (33) wil[illegible]ly contradict the requirement tha[illegible] Π. To avoid that difficulty, we'l[illegible]hat

$$\Pi_0 = 0. \tag{34}$$

Then $K_0(1) = 0$ and there is no need for the leading term ℓ_0 of the boundary layer correction. The case $\Pi_0 \neq 0$ deserves additional study. We note that additional complications occurred in earlier singular perturbation problems with terminal cost penalty [cf. O'Malley and Kung (1975) and Glizer and Dmitriev (1975)]. Likewise, we note that although the Riccati gain for the example of Womble et al. (1976) is asymptotically an additive function of t and σ, the approximate boundary layer correction is not simply a power series in ϵ, but a generalized asymptotic expansion [cf. Olver (1974)].

With $\Pi_0 = 0$, we take

$$\ell_0 = 0. \tag{35}$$

Then, the coefficient of ϵ^2 in (31) implies that ℓ_1 must be a decaying solution of

$$\begin{aligned} \ell_{1\sigma} = &-\ell_1B_2R_2^{-1}B_2'\ell_1 - \ell_1B_2R_2^{-1}B_2'K_1 \\ &- K_1B_2R_2^{-1}B_2'\ell_1 \end{aligned} \tag{36}$$

with

$$\begin{aligned} L_0(0) &\equiv R_2^{-1/2}(1)B_2'(1)\ell_1(0) \\ &= -R_2^{-1/2}(1)B_2'(1)K_1(1) \end{aligned}$$

known in terms of

$$C_0(1) \equiv R_2^{-1/2}(1)B_2'(1)K_1(1)B_2(1)R_2^{-1/2}(1) > 0$$

[cf. (21) and (22)]. We're thereby provided with a unique decaying solution

$$\begin{aligned} &\ell_1(\sigma) \\ &= -2L_0(0)(I + e^{2C_0(1)\sigma})^{-1}C_0^{-1}(1)L_0'(0). \end{aligned} \tag{37}$$

Further terms follow termwise as solutions of linearized equations. The needed initial value $P_2'(1)\ell(0,\epsilon)$ is obtained termwise by matching with the outer solution $K(1,\epsilon)$. We note that the positive definiteness of $B_2'QB_2$ is critical in obtaining the stability needed for the boundary layer correction $\ell(\sigma,\epsilon)$. The bifurcation aspect recurs since $P_2'(1)\ell$ is determined via differential equations, while $P_1'(1)\ell$ follows from the decay of ℓ. More details for the $m_1 = 0$ problem are contained in O'Malley (1976). A more complicated analysis would be required if $B_2'QB_2 = 0$ while $B_{21}'QB_{21} > 0$ [cf. O'Malley and Jameson (1977)] and more impulsive controls would result in "unbounded peaking" of the initial state [cf. Francis and Glover (1977)].

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